Domain decomposition for full-waves simulation in cold plasma

Takashi Hattori, Simon Labrunie, Jean Rodolphe Roche, Pierre Bertrand

Domain decomposition for Full-Wave simulation in a tokamak plasma

Takashi Hattori[1,*], **S. Labrunie**[1], **J.R. Roche**[1], **P. Bertrand**[2]
[1] Institut Elie Cartan de Lorraine, Université de Lorraine
[2] Institut Jean Lamour, Université de Lorraine
*Email: takashi.hattori@univ-lorraine.fr

Abstract

The aim of this work is to develop a numerical method for the full-wave simulation of electromagnetic wave propagation in a plasma. The propagation and the absorption of lower hybrid (LH) electromagnetic waves is a powerful method to generate current in tokamaks by Landau wave particle resonance. Full-wave calculations of the LH wave propagation is a challenging issue because of the short wave length with respect to the machine size. We propose a Fourier finite element method for solving the Maxwell equations based on a mixed augmented variational formulation. In order to develop a parallel version of the simulation and consider non homogenous plasma response, a nonoverlapping domain decomposition approach is presented.

Introduction

Let the domain Ω be a torus (tokamak plasma volume) with strong external time-invariant magnetic field $\boldsymbol{B}_{ext}$. We study a second order partial differential equation for the time-harmonic electric field $\boldsymbol{E}$ arising from Maxwell equations:

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \frac{\omega^2}{c^2}\underline{\boldsymbol{K}}\boldsymbol{E} &= \boldsymbol{f} \quad \text{in } \Omega, && (1)\\ \operatorname{div}(\underline{\boldsymbol{K}}\boldsymbol{E}) &= g \quad \text{in } \Omega && (2) \end{aligned}$$

where $\omega > 0$ is the excited wave frequency and c denotes the speed of light in free space. The plasma response is described by the matrix $\underline{\boldsymbol{K}}$, in Stix frame (third coordinate parallel to $\boldsymbol{B}_{ext}$). It includes a cold plasma approximation of the relative dielectric permittivity tensor and Landau damping:

$$\underline{\boldsymbol{K}}(\boldsymbol{x}) = \begin{pmatrix} S(\boldsymbol{x}) & -iD(\boldsymbol{x}) & 0 \\ iD(\boldsymbol{x}) & S(\boldsymbol{x}) & 0 \\ 0 & 0 & P_L(\boldsymbol{x}) \end{pmatrix}$$

Expressions of the entries S, D and P_L involve plasma frequencies, cyclotron frequencies of each species (ion and electron) and also the collision frequency. In general, the matrix $\underline{\boldsymbol{K}}$ is complex-valued and non-hermitian. Let Γ be the boundary of the domain Ω and $\Gamma_A \subset \Gamma$ be an antenna on the tokamak, then several boundary conditions are possible:

$$\begin{aligned} &\text{Neumann:} && \mathbf{curl}\,\boldsymbol{E} \times \boldsymbol{n} = i\omega\mu_0 \boldsymbol{j}_s \quad \text{on } \Gamma_A \\ &\text{Dirichlet:} && \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{E}_A \times \boldsymbol{n} \quad \text{on } \Gamma_A. \end{aligned}$$

On the other part of the boundary $\Gamma_C = \Gamma \setminus \Gamma_A$, we assume a perfectly conducting condition:

$$\boldsymbol{E} \times \boldsymbol{n} \;=\; \mathbf{0} \quad \text{on } \Gamma_C.$$

1 Finite element method

1.1 Variational formulation and well-posedness

Taking the divergence condition (2) as constraint, we use a mixed augmented variational formulation (MAVF) [3], which gives rise to a $\boldsymbol{H}^1$ conforming variational space, $\boldsymbol{X}_N(\underline{\boldsymbol{K}}, \Omega) := \boldsymbol{H}_0(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\operatorname{div}\underline{\boldsymbol{K}}, \Omega)$. We obtain the following variational formulation of the Dirichlet problem :
Find $(\boldsymbol{E}, p) \in \boldsymbol{X}_N(\underline{\boldsymbol{K}}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a_s(\boldsymbol{E}, \boldsymbol{F}) + \overline{b(\boldsymbol{F}, p)} &= L_s(\boldsymbol{F}) \quad \forall \boldsymbol{F} \in \boldsymbol{X}_N(\underline{\boldsymbol{K}}, \Omega) \\ b(\boldsymbol{E}, q) &= l(q) \quad \forall q \in L^2(\Omega). \end{aligned}$$

where

$$\begin{aligned} a_s(\boldsymbol{E}, \boldsymbol{F}) &:= (\mathbf{curl}\,\boldsymbol{E} \mid \mathbf{curl}\,\boldsymbol{F}) - \frac{\omega^2}{c^2}(\underline{\boldsymbol{K}}\boldsymbol{E} \mid \boldsymbol{F}) \\ &\quad + s(\operatorname{div}\underline{\boldsymbol{K}}\boldsymbol{E} \mid \operatorname{div}\underline{\boldsymbol{K}}\boldsymbol{F}) \\ L_s(\boldsymbol{F}) &:= (\boldsymbol{f} \mid \boldsymbol{F}) + s(g \mid \operatorname{div}\underline{\boldsymbol{K}}\boldsymbol{F}) \\ b(\boldsymbol{E}, q) &:= (\operatorname{div}\underline{\boldsymbol{K}}\boldsymbol{E} \mid q) \\ l(q) &:= (g \mid q), \end{aligned}$$

with parameter $s \in \mathbb{C}$. Here, $(\cdot \mid \cdot)$ denotes the standard L^2 inner product in Ω.

The well-posedness of the considered formulation follows from the Babuska-Brezzi theorem. Thanks to spectral properties of the dielectric tensor, the sesquilinear form a_s is coercive if $\Re(s) > 0$ and $\Im(s) \leq 0$.

1.2 Dimension reduction and discretization

The 3D problem can be reduced to a series of 2D one by using cylindrical coordinates (R, Z, ϕ) and by expanding all functions $f(R,Z,\phi)$ as Fourier series in the angular coordinate ϕ

$$f(R,Z,\phi) = \frac{1}{\sqrt{2\pi}} \sum_{\nu\in\mathbb{Z}} f_\nu(R,Z) e^{i\nu\phi}$$

where the coefficients $f_\nu(R,Z)$ are defined on a cross section of Ω [4]. Then the sesquilinear forms of the variational formulation can be written as sum of modal forms

$$a_s(\boldsymbol{u},\boldsymbol{v}) = \sum_{\nu\in\mathbb{Z}} a_{s,\nu}(\boldsymbol{u}_\nu,\boldsymbol{v}_\nu), \quad b(\boldsymbol{v},p) = \sum_{\nu\in\mathbb{Z}} b_\nu(\boldsymbol{v}_\nu,p_\nu)$$

The modal variational formulation is then discretized using a Taylor-Hood P_2-iso-P_1 finite element.

1.3 Numerical results

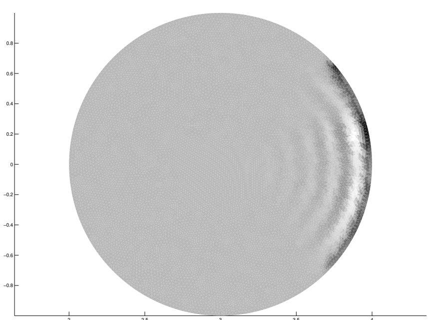

Figure 1: Real part of a component of the electric field for $\omega = \omega_{LH} = 1.3 \times 10^{10}$ rad/s

2 Domain decomposition

Consider a nonoverlapping decomposition $\overline{\Omega} = \bigcup_k \overline{\Omega_k}$. In the domain decomposition method considered here, we solve the original problem in each subdomain Ω_i; the equivalence with the one-domain formulation is obtained by continuity conditions

$$[\boldsymbol{E}\times\boldsymbol{n}]_{\Sigma_{ij}} = 0 \quad \text{and} \quad [\underline{\boldsymbol{K}}\boldsymbol{E}\cdot\boldsymbol{n}]_{\Sigma_{ij}} = 0 \qquad (3)$$

which ensure the $\boldsymbol{X}(\underline{\boldsymbol{K}},\Omega)$ regularity of the electric field and

$$[\mathbf{curl}\,\boldsymbol{E}\times\boldsymbol{n}]_{\Sigma_{ij}} = 0, \qquad (4)$$

which implies that the one-domain formulation holds in the sense of distributions. We have denoted, as usual, $[f]_{\Sigma_{ij}}$ the jump of a quantity f across the interface $\Sigma_{ij} = \overline{\Omega_i}\cap\overline{\Omega_j}$. The conditions (3) are dualized by introducing the associated Lagrange multipliers $\boldsymbol{\lambda}_{ij} \in \boldsymbol{H}^{1/2}(\Sigma_{ij})$, while (4) is treated as a natural condition. The existence and uniqueness of the solution $(\boldsymbol{E}_i, p_i, \boldsymbol{\lambda}_{ij})$ to the multidomain formulation was proved and :

$$\boldsymbol{E}_i = \boldsymbol{E}_{|\Omega_i} \quad \text{and} \quad p_i = p_{|\Omega_i}$$

where $(\boldsymbol{E},p)$ is the solution to the one-domain formulation.

The full linear system involving all subdomains (the *outer system*) is a generalized saddle-point problem:

$$\begin{pmatrix} \underline{\boldsymbol{Q}} & \underline{\boldsymbol{G}}^H \\ \underline{\boldsymbol{G}} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} \\ \lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{F} \\ 0 \end{pmatrix} \qquad (5)$$

where $\underline{\boldsymbol{Q}}$ is a block sparse non-hermitian matrix. Each block corresponds to a problem in one subdomain. The sparse matrix $\underline{\boldsymbol{G}}$ expresses the interactions between subdomains. The outer system (5) is solved using a preconditioned GMRES algorithm. The inner problem on each subdomain is also a generalized saddle-point problem, and is solved using a direct method.

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